PRESERVICE TEACHERS' UNDERSTANDING OF TWO-VARIABLE INEQUALITIES: APOS THEORY

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This study investigated preservice secondary mathematics teachers' difficulties with and understanding of graphs of two-variable inequalities with the lens of the APOS theory. The analysis showed that most teachers understood inequality graphs as actions, at best, and used informal rules to construct graphs. They also had difficulty justifying graphs, such as overgeneralization of linear inequality graphs, lack of understanding of the meaning of solutions and the Cartesian Connection, and inability to transfer algebraic/verbal to geometric rays. This study suggests two methods for the visualization of inequality graphs, which uses the concepts of variable and parameter.

Keywords: Inequality, Graph, Cartesian Connection, APOS theory

Inequalities is an important topic in mathematics due to its relations to equations and functions as well as its applications to real-life situations. Research on inequalities has been, however, under-represented to date despite calls for more research from the mathematics education community (Boero & Bazzini, 2004; Vaiyavutjamai & Clements, 2006). In the case of one-variable inequalities, studies have suggested that many of students' difficulties are associated with the properties of inequalities or the meaning of inequalities. For instance, students multiply a negative number to an inequality without changing the direction of the inequality sign (Kroll, 1986), overgeneralize the Zero Product Property to inequalities (Kroll, 1986), solve inequality problems by solving parallel equations (Almog & Ilany, 2012; Blanco & Garrote, 2007; Sfard, 1994; Tall, 2004; Vaiyavutjamai & Clements, 2006), and fail to interpret "solve" as finding values that make inequalities true (Blanco & Garrote, 2007; Frost, 2015).

In the case of two-variable inequalities, an article by Switzer (2014) describes how a guided discovery approach helps students understand the meaning/graph of a linear inequality in a specific form. Yet due to a lack of in-depth research on the topic, not much is known about what issues are involved in understanding various kinds of two-variable inequalities. In this paper, I study this under-examined area with the research question: What are the difficulties and understandings in constructing and justifying graphs of two-variable inequalities?

Theoretical Framework

This study utilizes the notion of the Cartesian Connection (CC) in equations and inequalities. The CC in equations is a relationship between an equation and its graph, which stipulates that 'a point is on the graph of a mathematical relation $R(x,y) = 0$ if and only if its coordinates satisfy the equation.' Past studies have reported that this seemingly trivial idea is rather complex, especially when individuals encounter the idea in non-typical tasks (Moon, Brenner, Jacob & Okamoto, 2013; Moschkovich, Schoenfeld, & Arcavi, 1993). 'CC in inequalities' is a newly minted terminology I created for this study. It means either 'a point is on the graph of the mathematical inequality $R(x,y) < 0$ if and only if its coordinates satisfy the inequality' or 'a point is on the graph of the $R(x,y) > 0$ if and only if its coordinates satisfy the inequality.'

On the other hand, as a graph of $R(x,y) = 0$ is a locus of points whose coordinates satisfy the equation, a graph of $R(x,y) < 0$ is a locus of points whose coordinates satisfy the inequality. As such, an understanding of the graph of $R(x,y) < 0$ may involve both localization and globalization: localization to attend to a pair of x and y values that satisfies $R(x,y) < 0$ and its representational transfer to a point in a plane, and globalization to attend to *all* such pairs and the graph of their corresponding points as a whole (see Even, 1998 for equations).

To transition from localization to globalization, one may need a mental mechanism that elevates actions and thoughts. Examples of such transitions include: a point-wise view to a global view (Monk, 1992), an operational view to a structural view (Sfard, 1991), and an action conception to a process conception in the APOS theory (Dubinsky & Harel, 1992). In this study, I use the APOS theory, which seems to best explain the phenomena involved in the visualization and understanding of graphs of two-variable inequalities.

An *action* is a mental or physical manipulation of objects that can transform one object into another. It can be a single-step action performed explicitly, including recalling memorized facts, or a multi-step action involving a number of steps without conscious control (Arnon et al., 2013; Cottrill et al., 1996). When actions are repeated and interiorized, the actions collectively become a mental *process* (Breidenbach et al., 1992). As such, an individual with a process conception may execute actions without explicitly running through each action.

Research has suggested that many individuals conceive graphs of one-variable functions or two-variable relations only as actions. They may plot and trace points to represent function equation graphs, but do not understand the existence of infinitude points or the continuity of the graphs (Kerslake, 1981; Leinhardt, Zaslavsky & Stein, 1990). They also overgeneralize linear function graphs to other graphs, even with their abilities to perform actions of transformations, from a table of x and y values to points in the plane (Presmeg & Nenduradu, 2005). Many calculus students also have, at best, an action view of two-variable functions. They can neither make connections between algebraic and geometric objects other than points nor explain cross sections of $y = x+z+3$ at fixed values of x (Trigueros & Martinez-Planell, 2010).

On the other hand, it has been shown that covariational reasoning (Weber & Thompson, 2014) is essential in the visualization of two-variable function graphs. A student who understands the graph of $y = x^2–2x$ as a collection of points on a plane (a process in one-variable functions) can visualize the graph of $y = z(x^2–2x)$ as a sweeping out of the $y = a(x^2–2x)$ graph, with a acting as a parameter and then as a variable (a process in two-variable functions). In comparison, another student who sees the $y = x^2–2x$ graph only as a shape is unable to visualize the graph of $y = z(x^2–2x)$.

Methodology

The participants were 15 preservice secondary teachers (referred as “teachers”) at a mid-sized state university in the Southeast region of the United States. Their mathematical backgrounds were varied, with four taking Precalculus, one taking Calculus I, and the other ten taking Calculus II or above. The participants were individually interviewed twice, for about one-and-a-half hours each time, in a form of semi-structured clinical interview. The interviews were recorded with a video camera and were transcribed. Their written responses were collected.

The interview items used for this study are the two questions included in the first interview.

Q1: (a) Find a solution of an inequality, $x+2y–32 < 0$.
(b) Represent all the solutions of the inequality above in the Cartesian plane.

Q2: (a) Find a solution of a system of inequalities, $y < x^2+1$ and $x^2+y^2 > 1$.
(b) Represent all the solutions of the system of inequalities above in the Cartesian plane.

During the interview, I asked the teachers why they represented the solutions in certain ways or why the x and y coordinates of all the points in their shaded regions satisfied the inequalities. I also asked the teachers to represent mathematical statements, in words or in algebraic forms, graphically in the plane when opportunities arose.

Genetic Decomposition

The genetic decomposition in APOS theory is a hypothetical model describing learners' mental structures and mechanisms for a certain concept (Arnon et al., 2013). For inequality graphs, I created a genetic decomposition by incorporating the action and process conceptions in the APOS theory and covariational reasoning (Weber & Thompson, 2014) to make up for the deficits in the current methods of teaching inequality graphs: the solution method and the ray method. The details follow.

In general, the solution method (ST) for the $y < f(x)$ graph involves two steps:

ST-1: A graph of $y = f(x)$ is drawn in a Cartesian plane.
ST-2: One or a few points are chosen from the plane, followed by the testing of the truth values of the points on $y < f(x)$. The graph is then determined as one of the two regions divided by the $y = f(x)$ graph in which points with true truth-values lie—normally shaded in gray or color (see, for example, David et al., 2011).

The first step of the ray method—see, for example, the CME Project, Algebra I (Cuoco et al., 2013)—is identical to ST-1. In the second step, however, the ray method uses verbal descriptions of rays to determine and justify inequality graphs.

> Try $x = 0$. The point on the line with x-coordinate 0 would be (0, 5). Any point on the vertical line with equation $x = 0$ with y-coordinate less than 5 is part of the solution set. Next try $x =$ 5. … It would take forever to write out the situation for every possible value of x, but you can see that any point that is below the line is part of the solution set. (p. 756).

The $y < 3x+5$ graph is then represented as the region under the $y = 3x+5$ graph, shaded in pink.

Both methods provide tools to construct and/or justify inequality graphs, yet they fall short of emphasizing the variable concept as a critical idea in inequality graphs, logically and/or graphically. To explain, the solution method yields inequality graphs by generalizing a few points with true test values to the entire region, with no justification provided; the ray method does not incorporate their verbal descriptions of rays in geometric representations, and inequality graphs are represented as static objects. The genetic decomposition, which I created and named the "concept of variable" (COV) method, addresses these issues. In COV, one simultaneously constructs and justifies graphs with the actions and process below:

- COV-1: One converts $R(x,y) = 0$ to its graph—a probable action as one generally depends on one's memory from the previous learning of equation graphs in this context.
- COV-2: One performs an action of assigning a value for the x variable in $R(x,y) < 0$, such as $x = k$, and performs algebraic treatments (Duval, 2006) on $R(k,y) < 0$ to solve for y. (It is possible that one fixes y as constant k and performs treatments on $R(x,k) < 0$)

- COV-3: One performs a conversion from the algebraic—$x = k$ and $R(k,y) < 0$—to a geometric object—line segment(s), ray(s), or line—depending on the nature of $R(k,y) < 0$.
- COV-4: With similar actions repeated for other values of x, one interiorizes the actions into a process and visualizes the $R(x,y) < 0$ graph as a sweeping out of segments, rays, and lines—similar to the two-variable function graphs through covariational reasoning (Weber & Thompson, 2014).

It is possible that one uses a combination of ST and COV to construct and justify inequality graphs: ST for construction and COV-related reasoning for justification. Although it is inefficient, some individuals may use this method, especially if they justify their graphs after they have constructed inequality graphs using ST.

For analysis, in addition to the genetic decomposition, I used an open coding strategy (Strauss, 1987) to examine teachers' difficulties with and understanding of inequalities, by using a priori codes such as action/process, algebraic/verbal/geometric, treatment/conversion, and equation/inequality. The initial coding showed patterns, and thus yielded categories and subcategories. I then performed the second stage of coding: reexamining and revising the prior codes, and at the same time performing an axial coding (Strauss, 1987) to focus on the categories and subcategories from the previous coding.

The results are presented in the next two sections. The first section focuses on teachers' understanding of inequalities with the lens of genetic decomposition, and the second their difficulties with inequalities in general.

Understanding Inequalities: Genetic Decomposition

Linear Inequality

Eleven teachers attempted to construct the $x+2y–32 < 0$ graph, with seven successfully doing so. All seven teachers with a correct inequality graph drew the line graph of $y = –x/2+16$, converted from $x+2y–32 = 0$ (ST-1, or equivalently COV-1), but none used the rest of ST or COV when constructing the inequality graph. Instead, five of them applied a rule, such as "less than is lower," and two simply shaded the lower part of the line with no explanation.

When justifying their graphs, two of the seven teachers with a correct inequality graph performed actions of checking the truth values of a few points under the line to $y<–x/2+16$, and then generalized the few points to the entire region. When asked why they included the entire region below the line, however, they could not articulate their reasoning. Another two teachers followed COV-2, making comments such as, "my y-values are never going to be bigger than 16 when $x = 0$," which were verbal representations of rays. However, when they were asked to represent their words in a plane, they claimed the entire region under the $y = –x/2+16$ graph as the geometric representation of their words, failing COV-3. The other three also failed to provide correct reasoning with various issues, including a shaky understanding of the Cartesian Connection, which I will explain in more detail in the next section.

Among the four teachers with an incorrect inequality graph, three teachers incorrectly drew a $y = 16–x/2$ graph (ST-1), with two using the "less is under" rule and one claiming the line graph itself as the inequality graph. One remaining teacher did not provide any explanation.

Parabolic Inequality

Seven teachers attempted to construct the $y < x^2+1$ graph, with six correctly representing the $y = x^2+1$ graph and three the $y < x^2+1$ graph. All three teachers with a correct $y < x^2+1$ graph first drew $y = x^2+1$ (ST-1) and then either performed ST-2 or used the rule, "less is under," to

construct a graph. Among the three teachers with an incorrect inequality graph, but with a correct $y = x^2+1$ graph (ST-1), one claimed that the upper part of the parabola was the inequality graph by guessing, and two claimed that the lower part of the $y = 1$ graph was the inequality graph, using the rule, "less is under." The remaining teacher represented $y = x^2+1$ as a curve similar to the $y = x^2$ graph (ST-1) and claimed that the curve itself was the inequality graph.

When justifying their graph, two of the six teachers who had a correct $y = x^2+1$ graph did not provide any explanation, and one teacher used the "less is under" rule. The other three teachers followed COV-2 by fixing x values and by saying such things as "I will make x equals 0, then you will get y is less than 1." However, all three incorrectly represented their words in a plane, failing COV-3. One represented his words as the half plane under the $y = 1$ graph, and the other two represented them as the entire region under the $y=x^2+1$ graph.

Circular Inequality

Seven teachers attempted to construct the $x^2+y^2 > 1$ graph, with five providing a correct graph and two an incorrect one. All five teachers with a correct inequality graph drew a $x^2+y^2 = 1$ graph (ST-1) and used a "greater is outside" rule to determine the inequality graph. The other two with an incorrect inequality treated $x^2+y^2 = 1$ into $y = \sqrt{1 - x^2}$ and sketched the latter as a curve similar to the $y = \sqrt{x}$ graph (ST-1). One teacher then shaded the upper part of the curve using the rule, "greater is upper," and the other claimed that the curve itself was the inequality graph.

When justifying the graph, one of the five teachers with a correct inequality graph used ST-2 as a justification; another teacher used a verbal description of a ray, "at $x = 2$, anything plugged in for y could be greater than 1" (COV-2), without performing the rest of COV; and another teacher did not provide any explanation at all. The other two teachers used an idea of $x^2+y^2 = k$ as a part of $x^2+y^2 > 1$—more generally, $R(x,y) = k$ as a part of $R(x,y) > 0$, implicitly utilizing the concept of parameter (COP), which I will explain in more detail in the Discussion and Conclusions section.

As for the two teachers who had a $y = \sqrt{x}$-like curve, one teacher provided reasoning by using a verbal description of a ray (COV-2), but represented his words as the entire region above the curve, failing COV-3. The other teacher was not asked to justify as she did not provide any justification for the linear and parabolic inequalities.

Difficulties with Inequalities

The teachers in general struggled to construct and justify inequality graphs, with fewer than half of them providing a correct graph—seven for the linear, five for the circular, and three for the parabolic inequality. The sources of their difficulties were various and intermingled, with some implicitly shown in the previous section. Below are the four most noteworthy characteristics of their difficulties.

Misunderstanding the Meaning of Solution

Some teachers did not understand the meaning of "solution" of an inequality and/or of a system of inequalities. Four teachers claimed that $y < -x/2+16$ (or $y <$ something) was a solution of $x+2y-32 < 0$. There were also three teachers who correctly interpreted the meaning of solution in a single inequality, but not in the system of inequalities. Instead of finding solutions of the system of inequalities, they found a solution for each inequality separately as if they had been given two independent inequality problems. Considering the fact that they had successfully found a solution of a system of equations—$y = x^2+2$ and $y = -2x$—in a different question, they

did not seem to understand the role of the *and* connective in the system of equations and inequalities.

Lacking Understanding of the Cartesian Connection

Four teachers successfully found the solutions of inequalities; however, they either falsely claimed the graphs of parallel equations themselves as the graphs of inequalities, or could not determine the inequality graphs due to their lack of understanding of CC. In one case, my questioning helped a teacher revise his thinking, reflect on the meaning of solution, and change his answer to a correct inequality graph. However, for the other three teachers, my questioning did not help them reorient their thinking.

> I: Why do you think everything on the line is a solution [of the inequality]?
> S: I see 1, 16 [in x+2y-32<0]. Oh no, that will be greater than 0. It is 0, 16. No, it wouldn't be because 0 is not less than 0. So, I guess when x is 1, y is 15.5, so 1+31 is 32 minus 32 is 0.
> I: Would everything on the line make [x+2y-32] zero?
> S: Well, we will see. 2 gives me 15. It seems like everything on this line will give me 0.
> I: So how does that help you determine the answer?
> S: It doesn't. I am confused.
> I: You are saying everything on the line is 0. But you can't determine what values will make it less than 0.
> S: Right.

As shown above, when asked whether the x and y coordinates of any point on the line would satisfy $x+2y–32 = 0$, the teacher was compelled to find another solution of $x+2y–32 = 0$, after having already found two such pairs meeting the condition, showing a lack of understanding of CC in equations. Further, when he had a pair of x and y values, $x = 1$ and $y = 16$, which made $x+2y–32$ greater than 0, he was unable to use the pair to determine the graph of $x+2y–32 < 0$; instead, he adjusted the x value to 0 to have a pair that satisfied $x+2y–32 = 0$, showing a lack of understanding of CC in inequalities.

Overgeneralization of Linear Inequality Graphs

The teachers' overgeneralizing behavior of linear inequality graphs appeared when two teachers represented the parabolic inequality graph. Despite having a correct $y = x^2+1$ graph, they shaded the half plane under the $y = 1$ graph instead of the region under the $y = x^2+1$ graph. One teacher provided valid reasoning, but an incorrect graph, by inserting the $y = 1$ graph. The other teacher simply said, "I don't know how to do this one," and shaded under the $y = 1$ graph. It was noteworthy that this behavior came from two higher performing teachers and, in particular, from a teacher whose reasoning was otherwise in the right direction.

Figure 1: Teachers' Representations of Rays in $y<x^2+1$

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Problems with Ray Conversions

Teachers also struggled with ray conversions. Of the five teachers who were asked to do the conversions, with two of them being asked twice, four teachers could not convert from algebraic/verbal rays to geometric rays. For the linear inequality, four teachers verbally described a ray, saying such things as "when $x = 31$, y is less than 0," but three of them falsely claimed the entire region under the $y = 16$-$x/2$ graph as the graph of their verbal ray. For the parabolic inequality, one teacher represented the statement, "$y < x^2+1$ when $x = 0$," as the entire region below the $y = 1$ graph instead of a ray on the y axis (Figure 1a). Another teacher represented the statement, "$y < x^2+1$ when $x = 3$," as a ray on the y-axis instead of a ray on the line, $x = 3$ (Figure 1b).

Discussion and Conclusions

This study examined preservice secondary teachers' abilities to construct and justify inequality graphs by applying the framework of the APOS theory (Dubinsky & Harel, 1992). The results showed that the vast majority of the teachers had, at best, an action conception of inequality graphs. They used rules, such as "less is lower" or "greater is upper," when constructing or justifying graphs, rather than reasoning relevant to the concept of variable or the Cartesian Connection. Even when they were able to implicitly use the concept of variable in their argument, they could not utilize the concept to construct or justify graphs, due partly to their lack of ability to convert their arguments of verbal/algebraic rays to geometric rays.

Their difficulties with inequalities were comparable to individuals' difficulties with inequalities and functions shown in other studies. Some teachers incorrectly found solutions of two-variable inequalities from parallel equations, similar to those who struggled with one-variable inequalities in earlier studies (Almog & Ilany, 2012; Blanco & Garrote, 2007; Frost, 2015). Some teachers falsely claimed that graphs of inequalities were the graphs of parallel equations, despite knowing the meaning of solutions in algebraic forms—similar to the individuals who could not make the Cartesian Connection in two-variable equations in other studies (Moon et al., 2013; Moschkovich et al., 1993). Some teachers also falsely represented the graph of a parabolic inequality with a linear inequality graph, similar to the teacher who represented the table of a non-linear function with a line graph in Presmeg and Nenduradu (2005).

Teachers also showed struggles that were comparable to those of college students in the case of two-variable functions. Many teachers knew the shapes of inequality graphs by memory, or by using rules, but did not understand inequality graphs at fixed values of x, similar to students who could not visualize the cross-sections of two-variable functions in other studies (Trigueros & Martinez-Planell, 2010; Weber & Thompson, 2014). As such, visualizing an inequality graph as a sweeping out of rays, line segments, or lines— similar to the way that a student visualized a two-variable function graph as a sweeping out of curves in Weber and Thompson (2015)—was out-of-reach for the teachers.

On the other hand, two teachers' reasoning shown in the circular inequality graph, which implicitly utilized the concept of parameter, inspired me to generate a new method for inequality graphs. In this concept of parameter (COP) method, inequality graphs can be constructed and justified as a sweeping out of curves or lines. To use the COP method, one first converts $R(x,y) = 0$ into a graph (COP-1, identical to COV-1). One then performs an action of assigning a value k to $R(x, y)$ by seeing $R(x,y) = k$, with $k > 0$, as part of $R(x,y) > 0$ (COP-2). One then converts

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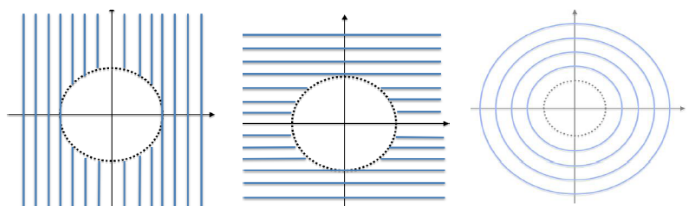
$R(x,y) = k$ to its graph, which is a curve (including lines) (COP-3). After repeating these actions for various k, one interiorizes these actions into a process and visualizes the inequality graph as a sweeping out of all curves (COP-4).

The results of the COV and COP methods are quite different when the graphs are shown as dynamic processes. While COV represents a graph as a collection of vertical or horizontal rays, lines, and line segments (Figure 2a & 2b, respectively), COP represents it as a collection of lines or curves (Figure 2c). As shown in the work by Moon (2019), the COV and COP methods have their own merits. As such, both methods deserve to be considered in the instruction of inequality graphs.



a b c

Figure 2: Graphs of $x^2+y^2>1$ through COV and COP Methods

I conclude this paper with a suggestion for those who contemplate implementation of the concepts of the variable and parameter methods. I believe that instructors should not use the methods as rote means to draw graphs of inequalities. Rather, they should focus on the concepts and ideas embedded in the methods: the meaning of inequalities, variable and parameter, the meaning of solutions, and the connection between algebraic and graphical inequalities, including the Cartesian Connection and ray connection.

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